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The spectrum and damping of waves in partially randomized multilayers

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Abstract. The spectrum and damping of waves in partially randomized multilayer structures are calculated. A method of calculation that was proposed and demonstrated earlier, for the model of a superlattice with a harmonic dependence of its material parameters along its axis in the initial state, is extended to the case of a multilayer structure (i.e., a superlattice with sharp interfaces). One- and three-dimensional random modulations of the period are considered, and the correlation function of the superlattice is derived as a series in which each term is a product of a harmonic and a monotonically decaying function. The law of decay of the correlation function is Gaussian for smooth inhomogeneities, and has different forms for one- and three-dimensional short-wavelength inhomogeneities. The spectrum and damping of waves in the superlattice described by this correlation function are found in the weak-coupling approximation in the vicinities of all of the odd Brillouin zone boundaries. Analytical dependences of the main characteristics of the spectrum and damping on the zone number n are obtained. The conditions for the closing of the gaps at the Brillouin zone boundaries are derived, and depend on the dimensionality of the inhomogeneities and the degree of their smoothness.

1. Introduction

Investigations of the spectrum of waves in partially randomized superlattices (multilayer structures) have been carried out very intensively in recent years. An even greater activity is dedicated to the kinetics of electrons in similar structures. Despite the fundamental differences between the physical problems, the mathematical approaches that are used for the development of a theory of partially randomized superlattices are analogous to a large degree for the waves as well as for the conduction electrons. Several such approaches now exist.

The modelling of the randomization by altering the order of successive layers of two different materials A and B (of different or the same thickness) is in wide use now. It is assumed that neither the parameters of the materials nor the layer thicknesses change when the system is randomized: only the periodicity ABAB $\cdots$ in the arrangement of the layers corresponding to the ideal superlattice is destroyed. The different versions of this model differ in the types of disruptions of the periodicity in the arrangement of the layers: in some versions the layers A and B are arranged according to the Fibonacci sequence rule; in others they form either partially correlated or totally uncorrelated random sequences. A number of important and interesting results have been derived with the help of these models in studies of electron dynamics [1–4], or the propagation of elastic [5, 6] and spin [7, 8] waves.

In several papers the study of wave propagation in a superlattice was conducted in the framework of a method which consists in the numerical modelling of the random deviations of the interfaces from their initial periodic arrangement [9–11].

The model of smooth interfaces, which is based on the introduction of a doubly periodic dependence of a physical parameter along the superlattice axis, has been used in another approach to the investigations of the consequences of the disruption of periodicity [12, 13]. In this approach the amplitude or phase of the main harmonic function which describes the initial ideal superlattice is modulated by another harmonic function. The periods of these two functions can be commensurate or incommensurate with each other. This model leads to extremely complicated and interesting spectra. A wide number of situations (connected, for example, with problems of the theory of quasicrystals) which arise when the disruptions of periodicity appear, are studied by this approach; the randomization of the spectrum of the system is only one of these situations.

One more approach to the description of partially randomized superlattices was proposed recently in references [14, 15]. This approach is based on the well known radio-physics model of the random modulation of the frequency or phase of a periodic radio signal [16, 17]. In reference [14] a brief outline of this approach is given for the case of spin waves in a superlattice whose period is modulated by a one-dimensional random function of a coordinate. In reference [15] a detailed description of the approach, and its extension to the cases of two- and three-dimensional random modulations of a superlattice, are presented. In this approach the correlation function of the superlattice is found analytically for each type of random modulation. The spectrum, damping, and other characteristics of the waves are calculated by this method for the model with a harmonic dependence of material parameters along the axis of the initial superlattice. The calculation is restricted to the first Brillouin zone. Such a model of a superlattice is sufficient for the demonstration of the application of this approach. However, it is very far from real superlattices of the type of multilayers which are widely investigated experimentally now. In the present paper this approach is extended to a multilayer system in which the dependence of material parameters in the initial state has the form of rectangular space pulses. It has been shown earlier [18] that for this type of superlattice the spectrum in all odd Brillouin zones can be studied by this method in the Bourret approximation [19], and not just the spectrum in the first zone as was done in reference [15].

The outline of this paper is as follows. In section 2 features of the application of the method suggested in reference [15] to the multilayer type of a superlattice are described. We also compare this method with a standard method of calculation of the spectrum of such a superlattice, and discuss drawbacks as well as advantages of both methods for the example of the ideal multilayer structure. In section 3 the correlation function of the multilayer structure is derived for different types of random modulations of the period of the structure. Because one of the main quantities appearing in the expression for the correlation function—the structure function of the random modulation—does not depend on the form of the superlattice, we can use all of the results for the structure function obtained in reference [15]. But one of the cases that is investigated in the present paper, namely the case of three-dimensional short-wavelength inhomogeneities, was not studied in reference [15] in sufficient detail. That is why considerable attention is given to the study of this case. In section 4 we investigate the spectrum and damping of a randomly modulated multilayer structure for the cases of one- and three-dimensional modulation. This investigation is carried out for all odd Brillouin zones. Dependences of the main parameters of the spectrum (the width of the gap at the nth Brillouin zone boundary, the damping of the waves) on the zone number n are obtained by analytical as well as numerical methods.

2. Methods of calculation; the spectrum of an ideal superlattice in the weak-coupling approximation

The method which we use in this paper for the investigation of the spectrum and damping of waves in partially disordered superlattices, like the majority of the methods which are used for this purpose, does not depend on the physical nature of the waves. For definiteness we consider here spin waves, but the main results are represented in a form that is valid in some approximations for elastic and electromagnetic waves as well.

We describe the dynamics of a ferromagnet by the Landau–Lifshitz equation

$$\dot{\boldsymbol{M}} = -g\left[\boldsymbol{M} \times \left(-\frac{\partial \mathcal{H}_m}{\partial \boldsymbol{M}} + \frac{\partial}{\partial \boldsymbol{x}} \frac{\partial \mathcal{H}_m}{\partial(\partial \boldsymbol{M}/\partial \boldsymbol{x})}\right)\right] \tag{1}$$

with the energy density

$$\mathcal{H}_m = \frac{1}{2}\alpha\left(\frac{\partial \boldsymbol{M}}{\partial \boldsymbol{x}}\right)^2 - \frac{1}{2}\beta(\boldsymbol{M}\cdot\boldsymbol{b})^2 - \boldsymbol{M}\cdot\boldsymbol{H}. \tag{2}$$

Here $\boldsymbol{M}$ is the magnetization, $\boldsymbol{H}$ is the magnetic field, g is the gyromagnetic ratio, α is the exchange parameter, β is the value of the magnetic anisotropy, and $\boldsymbol{b}$ is the direction of the anisotropy axis.

In homogeneous matter all of these parameters are constants characterizing the material. In an inhomogeneous medium they become random or regular (e.g., periodic) functions of the coordinates. We consider the consequences of such coordinate dependences of material parameters for wave propagation for the example of spin waves in a ferromagnet in which only the value of the magnetic anisotropy β depends on $\boldsymbol{x}$. Such an anisotropy can be represented in the form

$$\beta(\boldsymbol{x}) = \beta[1 + \gamma\rho(\boldsymbol{x})] \tag{3}$$

where β is the average value of the anisotropy, $\gamma = \Delta\beta/\beta$ is its relative rms fluctuation, and $\rho(\boldsymbol{x})$ is a centralized ($\langle\rho\rangle = 0$) and normalized ($\langle\rho^2\rangle = 1$) function of coordinates. For a random function $\rho(\boldsymbol{x})$ the angle brackets here denote either averaging over random realizations or spatial averaging. Both operations are equivalent according to the ergodicity principle for homogeneous random fields. For regular functions $\rho(\boldsymbol{x})$ the angle brackets denote spatial averaging.

Choosing $\boldsymbol{H}$ and $\boldsymbol{b}$ to be directed along the z-axis, performing the usual linearization of equation (1), and taking $M_x, M_y \propto \exp(\mathrm{i}\omega t)$, we obtain the following equation for the circular projection $m^+ = M_x + \mathrm{i}M_y$:

$$\nabla^2 m^+ + (\nu - \varepsilon\rho(\boldsymbol{x}))m^+ = 0. \tag{4}$$

In writing equation (4) we have introduced the notation

$$\nu = \frac{\omega - \omega_0}{\alpha g M} \qquad \varepsilon = \frac{\gamma\beta}{\alpha} \tag{5}$$

where $\omega_0 = g(H + \beta M)$. In the scalar approximation both the spectrum of elastic waves in a medium with an inhomogeneous density and the spectrum of electromagnetic waves in a medium with an inhomogeneous dielectric permeability are also described by this equation with redefinitions of the parameters. For elastic waves we have

$$\nu = (\omega/v)^2 \qquad \varepsilon = \nu\gamma_u \tag{6}$$

where γ_u is the rms fluctuation of the density of the material and v is the wave velocity. For an electromagnetic wave we have

$$\nu = \varepsilon_e(\omega/c)^2 \qquad \varepsilon = \nu\gamma_e \tag{6a}$$

where ε_e is the average value of the dielectric permeability, γ_e is its rms deviation, and c is the speed of light. Equation (4) becomes more complicated when inhomogeneities of the elastic modulus, of the exchange parameter, or of the magnetization are considered: terms of the form $(\nabla m^+)(\nabla\rho)$ appear in the equation in these cases. Inhomogeneities of the direction of the anisotropy axis also complicate the equation because they lead to the appearance of a stochastic magnetic structure in a ferromagnet, which interacts with spin waves [20]. In this paper we do not concern ourselves with such cases.

Let us consider an ideal multilayer structure which consists of periodically alternating layers of two materials with different physical properties—in our case, in the value of the magnetic anisotropy. Let the thickness of the interfaces between the layers go to zero, i.e., let us consider the model of sharp interfaces. It is well known that the exact solution of equation (4) can be found for this model. Indeed, equation (4) becomes an equation with constant coefficients in each material, which has a solution in the form of plane waves. Matching these solutions at the boundaries between layers, and using the periodicity conditions, one can obtain for the simplest case, where the thicknesses of the two materials are equal to each other, the following transcendental equation (see, for example, [21]):

$$\cos\frac{2\pi k_z}{q} = \cos\alpha_+\cos\alpha_- - \frac{\nu_1}{\sqrt{\nu_1^2-\varepsilon^2}}\sin\alpha_+\sin\alpha_- \tag{7}$$

where $\alpha_\pm = (\pi/q)\sqrt{\nu_1\pm\varepsilon}$, $\nu_1 = \nu-\kappa^2$, $\kappa^2 = k_x^2+k_y^2$, $\boldsymbol{k}$ is the wave vector, and $\boldsymbol{q}$ is the vector of the reciprocal superlattice ($|\boldsymbol{q}| \equiv q = 2\pi/l$, where l is the period of the superlattice).

For $\varepsilon/\nu_1 \ll 1$ equation (7) can be expanded as a power series in this parameter:

$$\begin{aligned}\sin\frac{\pi}{q}(\sqrt{\nu_1}-k_z)\sin\frac{\pi}{q}(\sqrt{\nu_1}+k_z) &= \frac{1}{8}\left(\frac{\varepsilon}{\nu_1}\right)^2[\alpha_0\sin 2\alpha_0 - 2\sin^2\alpha_0] \\ &+ \frac{1}{16}\left(\frac{\varepsilon}{\nu_1}\right)^4\left[\frac{1}{2}\alpha_0^2\left(\frac{3}{2}+\sin^2\alpha_0\right)+\frac{9}{8}\alpha_0\sin 2\alpha_0 - 3\sin^2\alpha_0\right]+\cdots\end{aligned} \tag{8}$$

where $\alpha_0 = (\pi/q)\sqrt{\nu_1}$.

For $\varepsilon\to 0$ the solutions of this equation are

$$\nu_{1n} = (k_z - nq)^2 \qquad n = 0, \pm 1, \pm 2, \ldots \tag{9}$$

The branches with $n\neq 0$ cross the main branch $\nu_1 = k_z^2$ at the crossing resonance points

$$k_{rn} = \frac{nq}{2} \qquad \nu_{rn} = \left(\frac{nq}{2}\right)^2 \tag{10}$$

which correspond to the boundaries of the Brillouin zones in the extended zone scheme.

To restrict ourself to the two-wave approximation in the vicinities of these crossing resonances we expand $\sin(\pi/q)(\sqrt{\nu_1}-k_z)$ on the left-hand side of equation (8) in the vicinity of the main branch $\sqrt{\nu_1} = k_z$, and $\sin(\pi/q)(\sqrt{\nu_1}+k_z)$ in the vicinity of the branches $\sqrt{\nu_1} = (nq-k_z)$, and then multiply both sides of the resulting equation by $(\sqrt{\nu_1}+k_z)[\sqrt{\nu_1}+(nq-k_z)]$. To obtain an equation for the eigenfrequencies in the vicinity of the odd Brillouin zone boundaries

$$n = 2m+1 \qquad m = 1, 2, 3, \ldots \tag{11}$$

only the first term on the right-hand side of equation (8), proportional to ε^2, needs to be kept, and the resonance values $k = k_{rn}$ and $\nu = \nu_{rn}$ have to be substituted into this term. For waves propagating along the z-axis ($k_z = k$, $\kappa = 0$) we have

$$(\nu - k^2)[\nu-(k-nq)^2] = \left(\frac{\Lambda}{2n}\right)^2 \tag{12}$$

where $\Lambda = 4\varepsilon/\pi$.

For obtaining the equation in the vicinity of the even Brillouin zone boundaries

$$n = 2m \qquad m = 1, 2, 3, \ldots \tag{13}$$

the next term of equation (8), proportional to ε^4, has to be taken into account, and the resonance values $k = k_{rn}$ and $\nu = \nu_{rn}$ have to be substituted into it. The term proportional to ε^2 vanishes at the resonance points, but we have to expand this term in the vicinity of these points. In this way we obtain the dispersion relation

$$(\nu - k^2)[\nu - (k - nq)^2] - \frac{1}{2}\left(\frac{\pi}{4}\right)^2\left(\frac{\Lambda}{k_{rn}}\right)^2(\nu - k_{rn}^2) - \frac{3}{16}\left(\frac{\pi}{4}\right)^4\left(\frac{\Lambda}{k_{rn}}\right)^4 = 0. \tag{14}$$

Equations (12) and (14) describe the well known effect of the removal of degeneracy and formation of gaps $\Delta\nu_n$ at the Brillouin zone boundaries, for the form of the superlattice considered:

$$\Delta\nu_n = \begin{cases} \dfrac{\Lambda}{n} & n = 2m + 1 \\ \left(\dfrac{\pi\Lambda}{2nq}\right)^2 & n = 2m. \end{cases} \tag{15}$$

One can see that the gaps at the boundaries of the even Brillouin zones are quantities of the next order in comparison with the gaps at the boundaries of the odd Brillouin zones, and they decrease proportionally to n^{-2} when n increases.

Let us consider now another method for obtaining the spectrum of a superlattice—the averaged Green function approach. This method takes into account from the very beginning that the parameter ε is small. That is a drawback of this method, but it has an advantage that is more important for us: the method permits studying superlattices with arbitrary dependences of ρ on $\boldsymbol{x}$, including a random dependence. By carrying out the Fourier transformation of equation (4), we obtain the integral equation satisfied by the transform $m_{\boldsymbol{k}}$:

$$(\nu - k^2)m_{\boldsymbol{k}} = \varepsilon \int m_{\boldsymbol{k}_1}\rho_{\boldsymbol{k}-\boldsymbol{k}_1}\,\mathrm{d}\boldsymbol{k}_1. \tag{16}$$

The eigenfrequencies of the waves described by equation (16) are determined by the poles of the Fourier transform of the average of the corresponding Green function. For equation (16) it has the form

$$G_{\nu,\boldsymbol{k}} = \frac{1}{\nu - k^2 - M_{\nu,\boldsymbol{k}}} \tag{17}$$

where the mass operator $M_{\nu,\boldsymbol{k}}$ is determined by a series in powers of ε.

In this paper we have restricted ourselves to considering only the first nonvanishing contribution in ε to the mass operator (the Bourret approximation):

$$M_{\nu,\boldsymbol{k}} = \frac{(2\pi)^3}{V}\varepsilon^2 \int \frac{\langle \rho_{\boldsymbol{k}-\boldsymbol{k}_1}\rho_{\boldsymbol{k}_1-\boldsymbol{k}}\rangle}{\nu - k_1^2}\,\mathrm{d}\boldsymbol{k}_1 \tag{18}$$

where V is the volume of the system. For any homogeneous random function the relationship

$$\langle \rho(\boldsymbol{k})\rho(\boldsymbol{k}')\rangle = S(\boldsymbol{k})\delta(\boldsymbol{k} - \boldsymbol{k}') \tag{19}$$

holds, where $S(\boldsymbol{k})$ is the spectral density of the random function $\rho(\boldsymbol{x})$, which is connected with the correlation function $K(\boldsymbol{r}) = \langle \rho(\boldsymbol{x})\rho(\boldsymbol{x} + \boldsymbol{r})\rangle$ by a Fourier transformation (the Wiener–Khinchin theorem):

$$K(\boldsymbol{r}) \equiv \langle \rho(\boldsymbol{x})\rho(\boldsymbol{x} + \boldsymbol{r})\rangle = \int S(\boldsymbol{k})\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\,\mathrm{d}\boldsymbol{k}. \tag{20}$$

Substituting equation (19) into equation (18), and taking into account that $\delta(0) = V/(2\pi)^3$, we obtain the general equation for the dispersion law of the averaged waves in the Bourret approximation in the form

$$\nu - k^2 = \varepsilon^2 \int \frac{S(\boldsymbol{k} - \boldsymbol{k}_1)\, \mathrm{d}\boldsymbol{k}_1}{\nu - k_1^2}. \tag{21}$$

Thus, all we need to know for the calculation of the spectrum of waves in this approximation is the correlation function of the superlattice. In deriving this correlation function we will follow the method which was suggested in reference [15], which is a generalization of the well known radio-physics model of the stochastic modulation of the frequency of a periodic radio-signal [16, 17] to the case of one-, two-, and three-dimensional inhomogeneities in a superlattice. In reference [15] this method has been applied to the model of an initial superlattice with harmonic dependences of its material parameters along the z-axis. We extend this approach here to a multilayer structure, i.e., to a superlattice whose material parameters depend on z in the form of rectangular spatial pulses in the initial state, and can be represented by a Fourier series. The randomization is taken account by introducing a random modulation u of the superlattice period. In the general case this modulation can be a function of all coordinates x, y, and z:

$$\rho(\boldsymbol{x}) = \frac{4}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^m}{p} \cos p\Big[q\big(z - u(\boldsymbol{x})\big) + \psi\Big] \tag{22}$$

where $p = 2m + 1$.

In the absence of disorder $\rho(\boldsymbol{x})$ has the form of rectangular spatial pulses. The stochastic properties of the function $\rho(\boldsymbol{x})$ have to be derived from the stochastic properties of the function $u(\boldsymbol{x})$ which characterizes, in the main, the inhomogeneity of the positions and structure of the interfaces. The latter function belongs to the class of inhomogeneous random functions with homogeneous increments. However, the random function $\rho(\boldsymbol{x})$, which depends on $u(\boldsymbol{x})$, is already homogeneous by virtue of its bounded amplitude, and can be characterized by the correlation function $K(\boldsymbol{r})$. As in reference [15] we have introduced a coordinate-independent random phase ψ, which is characterized by a uniform distribution in the interval $(-\pi, \pi)$. Hence, we consider an ensemble of random realizations of the superlattice. The condition of ergodicity is now satisfied for the function $\rho(z)$: the spatial average is equal to the ensemble average, and we can use the correlation theory, i.e. equations (19), (20), and (21), even in the case where $u = 0$.

The product of the functions $\rho(\boldsymbol{x})$ and $\rho(\boldsymbol{x} + \boldsymbol{r})$ can be represented in the form

$$\begin{aligned}\rho(\boldsymbol{x}+\boldsymbol{r})\rho(\boldsymbol{x}) = {} & \frac{8}{\pi^2} \sum_{m=0}^{\infty} \sum_{m'=0}^{\infty} \frac{(-1)^{m+m'}}{pp'} \\ & \times \Big\{ \cos q\Big[pr_z - p'u(\boldsymbol{x}) + pu(\boldsymbol{x}+\boldsymbol{r}) + (p - p')(z + \psi/q)\Big] \\ & + \cos q\Big[pr_z - p'u(\boldsymbol{x}) - pu(\boldsymbol{x}+\boldsymbol{r}) + (p + p')(z + \psi/q)\Big]\Big\}\end{aligned} \tag{23}$$

where $p' = 2m' + 1$. The second summand in the braces vanishes after averaging over the phase ψ. The terms with $p' \neq p$ in the first summand vanish as well, and after this averaging we have

$$\langle \rho(\boldsymbol{x}+\boldsymbol{r})\rho(\boldsymbol{x})\rangle_\psi = \frac{8}{\pi^2} \sum_{m=0}^{\infty} \frac{1}{p^2} \cos p(qr_z + \chi) \tag{24}$$

where

$$\chi(\boldsymbol{x}, \boldsymbol{r}) = q\Big[u(\boldsymbol{x}+\boldsymbol{r}) - u(\boldsymbol{x})\Big]. \tag{25}$$

Averaging equation (24) over χ with a Gaussian distribution function for χ, we obtain a general expression for the correlation function in the form

$$K(\boldsymbol{r}) = \frac{8}{\pi^2}\sum_{m=0}^{\infty}\frac{1}{p^2}\cos pqr_z \exp\left[-\frac{p^2}{2}Q(\boldsymbol{r})\right] \tag{26}$$

where

$$Q(\boldsymbol{r}) = q^2\left\langle\Big[u(\boldsymbol{x}+\boldsymbol{r}) - u(\boldsymbol{x})\Big]^2\right\rangle \tag{27}$$

is the dimensionless structure function of the random displacements $u(\boldsymbol{x})$.

For the special case of an ensemble of the ideal initial superlattices for which $u = 0$, $Q = 0$, we obtain the correlation function in the form

$$K(r) = \frac{8}{\pi^2}\sum_{m=0}^{\infty}\frac{1}{p^2}\cos pqr_z \tag{28}$$

and the spectral density corresponding to it in the form

$$S(\boldsymbol{k}) = \frac{4}{\pi^2}\delta(k_x)\delta(k_y)\sum_{m=0}^{\infty}\frac{1}{p^2}\Big[\delta(k_z - pq) + \delta(k_z + pq)\Big]. \tag{29}$$

Substituting this expression into equation (21) we obtain the equation for the eigenfrequencies of the ideal superlattice

$$\nu - k^2 = \frac{\Lambda^2}{4}\sum_{m=-\infty}^{\infty}\frac{1}{p^2}\frac{1}{\nu - (\boldsymbol{k} - p\boldsymbol{q})^2}. \tag{30}$$

One can see that the two-wave approximation corresponds to the neglect in this equation of all terms of the series except one $(p = n)$. In fact, equation (12) follows from equation (30) in this case.

To obtain equation (14), which corresponds to the even Brillouin zones, one can take into account the next term in the expansion of the mass operator (18). We do not do that in this paper, and restrict ourselves to considering only the odd crossing resonances, for which the gaps have the largest values and decrease most slowly with increasing n.

It should be noted that the series in equation (30) can be summed, and we obtain

$$\nu - k^2 = \left(\frac{\pi}{4}\right)^3\frac{\Lambda^2}{q\sqrt{\nu_1}}\left[\frac{1}{\zeta_-}\left(1 - \frac{1}{\zeta_-}\tan\zeta_-\right) + \frac{1}{\zeta_+}\left(1 - \frac{1}{\zeta_+}\tan\zeta_+\right)\right] \tag{31}$$

where

$$\zeta_\pm = \frac{\pi}{2q}\left(\sqrt{\nu - \kappa^2} \pm k_z\right). \tag{32}$$

This transcendental equation, which corresponds to the ‘exact’ Bourret approximation (30), contains only the second power of Λ explicitly. But the expansion of its solutions can contain all powers of this parameter. The same is true for equation (8) when only the first term on the right-hand side is taken into account. The solutions of these equations must not coincide identically. But there is such a coincidence for the odd Brillouin zones through terms of second order in Λ at least: both equations give the identical approximate equation (12) in the vicinity of the Brillouin zone boundaries for odd values of n.

3. The correlation function of a randomized multilayer structure

The general expression for the correlation function (26) derived here differs significantly from the corresponding general expression obtained in reference [15] for the model of the superlattice with a harmonic dependence of material parameters (see equation (I.16)) [22]. But the structure function $Q(\boldsymbol{r})$ which appears in equation (26) does not depend on the form of the function which describes the initial ideal superlattice. So, we can use here the results which have been obtained for $Q(\boldsymbol{r})$ in reference [15], amplifying and correcting them if necessary.

The structure function $Q(\boldsymbol{r})$ is connected through equation (I.22) with the spectral density $S_\phi(\boldsymbol{k})$ of the uniform random function $\phi(\boldsymbol{x}) = \nabla u(\boldsymbol{x})$:

$$Q(\boldsymbol{r}) = 2q^2 \int \frac{\mathrm{d}\boldsymbol{k}}{k^2}\, S_\phi(\boldsymbol{k})(1 - \cos \boldsymbol{k}\cdot\boldsymbol{r}). \tag{33}$$

Correlation properties of the function $\phi(\boldsymbol{x})$ can be modelled by some standard correlation function $K_\phi(\boldsymbol{r})$. One of the main results of reference [15] is that the structure function $Q(\boldsymbol{r})$ and, consequently, the correlation function of the superlattice $K(\boldsymbol{r})$ do not depend on the form of the modelling function $K_\phi(\boldsymbol{r})$, for the limiting cases of short-wavelength and long-wavelength inhomogeneities (see equations (I.23)–(I.32)). This statement is valid also for the multilayer type of superlattice, but the determination of these limiting cases becomes more complicated. We obtain the expression for the correlation function in the one-dimensional case in the approximate form

$$K(\boldsymbol{r}) = \frac{8}{\pi^2} \sum_{m=0}^{\infty} \frac{1}{p^2} \cos pqr_z \Phi_p \tag{34}$$

where

$$\Phi_p = \begin{cases} \exp(-p^2 k_{c1}^2 r_z^2/2) & p \gg p_0 \\ \exp(-p^2 k_{c2} r_z) & p \ll p_0. \end{cases} \tag{35}$$

Here $k_{c1} = \sigma q$, $k_{c2} = (\sigma q)^2/k_\parallel$, and $p_0 = k_\parallel/\sigma q$; σ is the rms fluctuation of the one-dimensional random function $\phi(z)$, and $k_\parallel$ is the correlation wavenumber of this function.

As for the superlattice with a harmonic initial dependence, we have a Gaussian decay of correlations for the case of smooth fluctuations of $u(z)$ and an exponential decay for short-wavelength fluctuations. But the demarcation line between the 'smooth' and 'short-wavelength' inhomogeneities now depends on the number p of the harmonic of the series. For the harmonics with $p \gg p_0$ the upper line of equation (35) holds, while for the harmonics with $p \ll p_0$ the lower one is valid. If $p_0 < 1$ we have Gaussian decay for all harmonics of the series.

In obtaining equation (35) we used equations (I.27) for $Q(r_z)$ in the limiting cases of large and small r_z. When the condition $p \gg p_0$ or $p \ll p_0$ is satisfied, the corresponding expression is approximately valid everywhere within the region of variation of r_z (see the justification for such an approach for $p = 1$ in reference [15]).

An analogous approach is valid for smooth inhomogeneities in the three-dimensional case as well. But for the short-wavelength inhomogeneities it cannot be used in the latter case because the corresponding approximate expression for $Q(\boldsymbol{r})$ diverges at $r \to 0$. In fact, let us consider the exact expression (I.30) for $Q(\boldsymbol{r})$ in the three-dimensional case:

$$Q(\boldsymbol{r}) = 2\left(\frac{\sigma q}{k_0}\right)^2 \left[1 - \frac{2}{k_0 r} + \left(1 + \frac{2}{k_0 r}\right)\exp(-k_0 r)\right] \tag{36}$$

where σ is the rms fluctuation of the three-dimensional random function $\phi(\boldsymbol{r})$ with isotropic correlation properties that are characterized by the correlation wavenumber k_0. (Note that there is misprint in equation (I.30); the correct expression, which was in fact analysed in reference [15], corresponds to equation (36) of the present paper.) The limiting cases of these expressions are

$$Q(\boldsymbol{r}) \approx \begin{cases} (\sigma q r)^2/3 & k_0 r \ll 1 \\ 2\left(\dfrac{\sigma q}{k_0}\right)^2\left(1 - \dfrac{2}{k_0 r}\right) & k_0 r \gg 1. \end{cases} \tag{37}$$

One can see that the correlation function $K(\boldsymbol{r})$ with $Q(\boldsymbol{r})$ determined by the lower line of equation (37) cannot be used everywhere within the region of variation of $\boldsymbol{r}$. For the case of three-dimensional short-wavelength inhomogeneities we will use here another approximation. Since $Q(\boldsymbol{r})$ is limited by the value of $2(\sigma q/k_0)^2$, we represent the exponent in equation (26) as a power series in $p^2 Q/2$ and restrict ourselves to the first power of this quantity. In so doing we use for $Q(\boldsymbol{r})$ the exact equation (36). As a result we can represent the correlation function for three-dimensional inhomogeneities in the form of equation (34), where

$$\Phi_p = \begin{cases} \exp(-p^2 k_{c3}^2 r^2/2) & p \gg p_0 \\ 1 - \left(\dfrac{p}{p_0}\right)^2\left[1 - \dfrac{2}{k_0 r} + \left(1 + \dfrac{2}{k_0 r}\right)\mathrm{e}^{-k_0 r}\right] & p \ll p_0. \end{cases} \tag{38}$$

Here $k_{c3} = \sigma q/\sqrt{3}$ and $p_0 = k_0/\sigma q$.

As in the one-dimensional case, the critical number p_0 divides the harmonics of the series (26) into two groups, with different approximate expressions for each of them. If $p_0 < 1$ we have Gaussian decay for all terms of the series.

Performing the Fourier transformation of the correlation function (34), where Φ_p is given by equation (35), we obtain the spectral density for the case of one-dimensional inhomogeneities in the form

$$S(\boldsymbol{k}) = \delta(k_x)\delta(k_y)\frac{8}{\pi^2}\sum_{m=-\infty}^{\infty}\frac{1}{p^2}S_p(k_z) \tag{39}$$

where

$$S_p(k_z) = \begin{cases} \dfrac{1}{\pi^2 k_{c1} p}\exp[-(k_z - pq)^2/2p^2 k_{c1}^2] & |p| \gg p_0 \\ \dfrac{k_{c2}}{2\pi}[p^4 k_{c2}^2 + (k_z - pq)^2]^{-1} & |p| \ll p_0. \end{cases} \tag{40}$$

Performing the Fourier transformation of the correlation function (34), where Φ_p is given by equation (38), we obtain the spectral density for the case of three-dimensional inhomogeneities. For $|p| \gg p_0$ we obtain

$$S(\boldsymbol{k}) = \frac{\sqrt{2}}{\pi^{7/2}k_{c3}^3}\sum_{m=-\infty}^{\infty}\frac{1}{p^5}\exp[-(\boldsymbol{k} - p\boldsymbol{q})^2/2p^2 k_{c3}^2] \tag{41}$$

while for $|p| \ll p_0$

$$S(\boldsymbol{k}) = \frac{4}{\pi^2}\sum_{m=-\infty}^{\infty}\left\{\left(\frac{1}{p^2} - \frac{1}{p_0^2}\right)\delta(\boldsymbol{k} - p\boldsymbol{q}) - \frac{k_0}{p_0^2(k_0^2 + |\boldsymbol{k} - p\boldsymbol{q}|^2)^2} + \frac{1}{\pi^2 p_0^2 k_0}\frac{1}{|\boldsymbol{k} - p\boldsymbol{q}|}\int_0^\infty (1 - \mathrm{e}^{-k_0 r})\sin(r|\boldsymbol{k} - p\boldsymbol{q}|)\,\mathrm{d}r\right\}. \tag{42}$$

4. The spectrum and damping of waves

Substituting the expressions for $S(\boldsymbol{k})$ obtained above into equation (21) and performing the integration we obtain equations for the spectrum of waves in the extended zone scheme. With the appearance of randomness in the medium the dispersion law $\omega(\boldsymbol{k})$ of the average waves becomes a complex function. There are two approaches to the analysis of such a law, depending on the formulation of the problem. If we consider a boundary-value problem on the spectrum and damping of standing waves in the absence of excitation, the wave vector $\boldsymbol{k}$ is a real quantity, while the frequency is complex: $\omega = \omega' + \mathrm{i}\omega''$, where $T = 1/\omega''$ determines the relaxation time of the standing waves. If we consider the problem of wave propagation for given initial conditions, the frequency is a real quantity, while the wave vector is complex: $\boldsymbol{k} = \boldsymbol{k}' + \mathrm{i}\boldsymbol{k}''$, and $1/k''$ determines the mean free path of the wave. We will use the first approach in the derivation and analysis of the complex dispersion laws.

4.1. Planar one-dimensional inhomogeneities

Substituting equation (39) into equation (21) we obtain the equation for the spectrum in the form

$$\nu - k^2 = \frac{\Lambda^2}{4}\left\{\sum_{|p|\ll p_0} F_p^{(1)}(\nu, k) + \sum_{|p|\gg p_0} F_p^{(2)}(\nu, k)\right\}. \tag{43}$$

The functions $F_p^{(1)}$ and $F_p^{(2)}$ in this equation are

$$F_p^{(1)} = \frac{\sqrt{\nu_1} - \mathrm{i}p^2 k_{c2}}{p^2\sqrt{\nu_1}}\,\frac{1}{(\sqrt{\nu_1} - \mathrm{i}p^2 k_{c2})^2 - (pq - k_z)^2} \tag{44}$$

$$F_p^{(2)} = \frac{1}{p^3 k_{c1}\sqrt{2\nu_1}}\left[D(u) + D(v) + \mathrm{i}\frac{\sqrt{\pi}}{2}(\mathrm{e}^{-u^2} + \mathrm{e}^{-v^2})\right] \tag{45}$$

where

$$D(x) = \mathrm{e}^{-x^2}\int_0^x \mathrm{e}^{t^2}\,\mathrm{d}t$$

is Dawson's integral, whose arguments u and v are given by

$$u = \frac{1}{\sqrt{2}pk_{c1}}[\sqrt{\nu_1} - |k_z - pq|]$$

$$v = \frac{1}{\sqrt{2}pk_{c1}}[\sqrt{\nu_1} + |k_z - pq|]. \tag{46}$$

The series from $-\infty$ to $+\infty$ on the right-hand side of equation (43) divides into two parts according to the inequalities that determine the different forms of $S_p(k)$ in equation (39). The terms with $|p| \sim p_0$ are absent on the right-hand side of equation (43) because their analytical forms are unknown to us. The complete equation (43) is very complicated for analytical analysis. But in the two-wave approximation we can describe the spectrum in the vicinity of each crossing resonance $k \sim k_{rn} = nq/2$ by using only the term of the series with $p = n$. For waves propagating along the z-axis ($k_z = k$, $\nu_1 = \nu$) we obtain in the case $|p| \ll p_0$ (short-wavelength inhomogeneities) the equation

$$(\nu - k^2)[(\sqrt{\nu} - \mathrm{i}n^2 k_{c2})^2 - (nq - k)^2] = \frac{\Lambda^2}{4n^2}\frac{\sqrt{\nu} - \mathrm{i}n^2 k_{c2}}{\sqrt{\nu}} \tag{47}$$

where $n = 2m + 1$.

Under the condition $nk_{c2}/k_r \ll 1$ we can neglect both the imaginary part of the coupling parameter and the shift of the crossing resonance point, and obtain the equation in the two-wave approximation in the form

$$(\nu - k^2)[\nu - \mathrm{i}n^3 G_2 - (nq - k)^2] = \left(\frac{\Lambda}{2n}\right)^2 \tag{48}$$

where $G_2 \approx k_{c2}q = \sigma q^2/p_0 = \sigma^2 q^3/k_\parallel$ is the damping parameter.

This equation has the complex solution at the nth Brillouin zone boundary

$$\nu^\pm = \nu_{rn} \pm \frac{1}{2}\Delta\nu_n + \mathrm{i}\xi_\pm \tag{49}$$

where

$$\Delta\nu_n = \sqrt{\left(\frac{\Lambda}{n}\right)^2 - (n^3 G_2)^2} \tag{50}$$

is the width of a gap, and

$$\xi_\pm = n^3 G_2/2 \tag{51}$$

is the damping of the wave.

The features of the behaviour of both the real $\nu(k)$ and imaginary $\xi(k)$ parts of the solution in the vicinity of the resonance points k_{nr} are the same as those at the first Brillouin zone boundary, which have been considered in reference [15]. But now the width of the gap and the condition for the gap to be opened depend on the number n of the zone. If the inequality

$$\frac{\Lambda}{n} > n^3 G_2 \tag{52}$$

is satisfied, the degeneracy is removed at the resonance point, the real parts of the dispersion curves $\nu'_\pm(k)$ separate, and a gap $\Delta\nu_n$ appears in the spectrum. In the opposite case the dispersion curve is continuous and has an inflexion at the resonance point.

All these results lead us to formulate the following rule of similarity: all characteristics of the spectrum and damping have an identical form in the vicinity of the boundaries of all Brillouin zones with zone number $n \ll p_0$ if we introduce an effective coupling parameter Λ_n and an effective damping G_n by

$$\Lambda_n = \frac{\Lambda}{n} \qquad G_{n2} = n^3 G_2. \tag{53}$$

Now we consider the case $|p| \gg p_0$ (long-wavelength inhomogeneities) in the two-wave approximation with $p = n$. For wave propagation along the z-axis we have the equation

$$\nu - k^2 = \frac{q\Lambda^2}{4G_1 n^3\sqrt{2\nu}} \left\{ D(u) + D(v) + \mathrm{i}\frac{\sqrt{\pi}}{2}(\mathrm{e}^{-u^2} + \mathrm{e}^{-v^2}) \right\} \tag{54}$$

where $G_1 = qk_{c1} = \sigma q^2$, and u and v are defined by equation (46) with $p = n$. For the first Brillouin zone boundary ($n = 1$) this equation has been analysed in reference [15] by numerical methods in the interval $0.2 < G_1/\Lambda < 0.8$. An analytical estimation of the solution for the case $G_1/\Lambda \ll 1$ has also been carried out. It was shown that there are significant differences between the behaviour of the solutions of equations (54) and (47) in the vicinity of $k_r = q/2$. If a monotonic decrease of the gap $\Delta\nu$ with increasing k_{c1} has been obtained for the exponential correlations, the increase of randomization for the Gaussian correlations leads at first to the increase of the gap and only then to its decrease and closing. There are also differences between the imaginary parts of the solutions of equations (54) and (47), especially in the case where $G_{1,2} \ll \Lambda$. At the resonance point the damping constants $\xi_\pm$ are always equal to each other, but for the exponential correlations they are both equal to $G_2/2$, while for the Gaussian correlations they may be much smaller than $G_1/2$.

Here we solve equation (54) numerically in a wider interval of damping variations. We analyse analytically more carefully the limiting case of small damping, and also find an approximate analytical solution in the limiting case when the gap becomes narrow in comparison with its initial value. This permits us to obtain an expression for the effective damping parameter G_{n1} for the case of long-wavelength inhomogeneities.

First of all we point out that the contributions of the two terms in equation (54) that depend on v are negligibly small in comparison with the terms depending on u. For small damping the argument $u \gg 1$, and we obtain the asymptotic expression for Dawson's integral using the Laplace method (see, e.g., reference [23]):

$$D(u) \approx \frac{1}{2u}\left(1 + \frac{1}{2u^2}\right). \tag{55}$$

In this approximation equation (54) can be represented in the form

$$u^4 - \frac{b}{2}(1 + \mathrm{i}\sqrt{\pi}u\mathrm{e}^{-u^2})u^2 - \frac{b}{4} = 0 \tag{56}$$

where $b = (\Lambda/2n^3G_1)^2$, and the imaginary term is small in comparison with unity. Solving this equation as a biquadratic equation in zero approximation, and taking into account the imaginary term in the next approximation, we obtain the expressions for the gap and damping

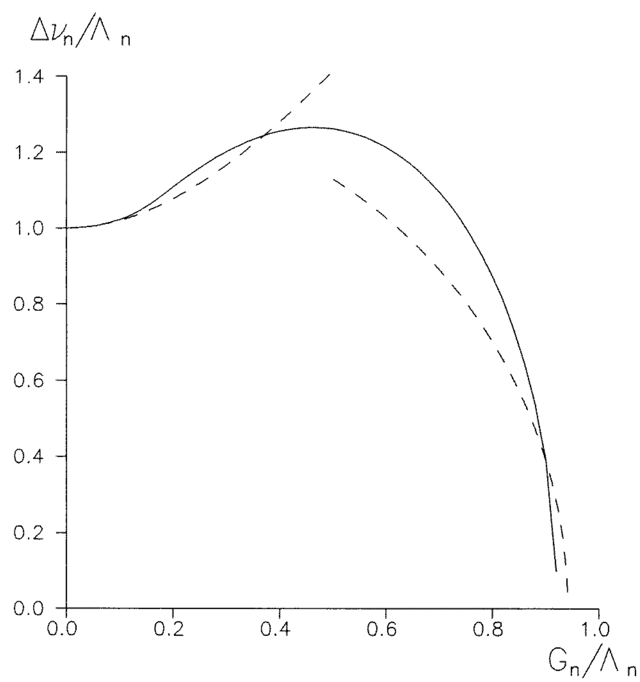


Figure 1. The dependence of the normalized width of the gap $\Delta\nu_n/\Lambda_n$ on the normalized value of the damping G_n/Λ_n caused by disorder for the Gaussian correlations of the inhomogeneities (solid curve). The approximations corresponding to equations (57) and (59) are shown by the dashed curves.

at the nth Brillouin zone boundary for $b \gg 1$:

$$\Delta\nu_n = \frac{\Lambda}{n}\left(1 + \frac{1}{b}\right)^{1/2} \tag{57}$$

$$\xi_\pm = \frac{1}{2}\left(\frac{\pi}{2}\right)^{1/2} G_1 n^2 (b-1) \exp\left[-\frac{1}{2}(b+1)\right]. \tag{58}$$

In the opposite limiting case, where the influence of the inhomogeneities is large enough, and the gap becomes narrow in comparison with its initial value, by using the approximate expression for $D(u)$ for $u \ll 1$, we obtain

$$\Delta\nu_n = \left[\sqrt{\pi}\left(\frac{\Lambda}{n}\right)^2 - 2(n^2 G_1)^2\right]^{1/2} \tag{59}$$

$$\xi_\pm = n^2 G_1/\sqrt{2} = n^2 \sigma q^2/\sqrt{2}. \tag{60}$$

The condition for the gap to be open is now determined by an inequality different to equation (52), namely

$$\frac{\Lambda}{n} > \left(\frac{2}{\sqrt{\pi}}\right)^{1/2} n^2 G_1. \tag{61}$$

One can see that the rule of similarly holds for the cases $n \gg p_0$ too, but the effective damping parameter now has another power in its dependence on n, namely $G_n = n^2 G_1$.

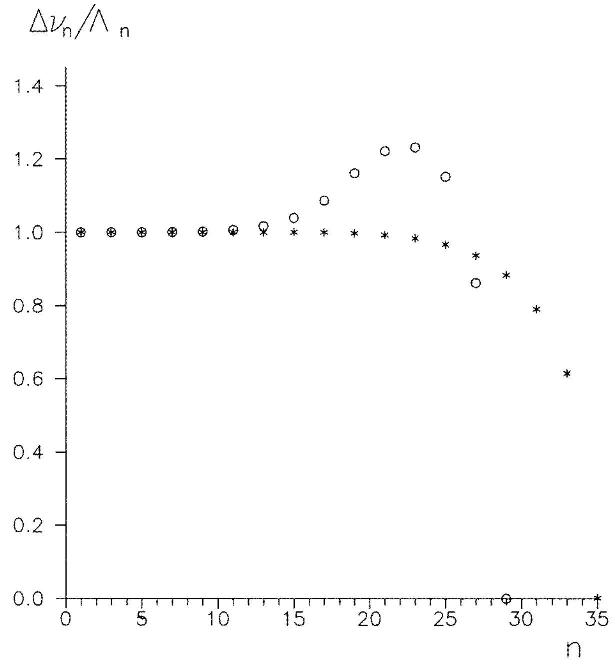
The dependence of $\Delta\nu_n$ on G_n which was obtained from the numerical solution of equation (54) is shown in figure 1 in the normalized coordinates $Y = \Delta\nu_n/\Lambda_n$ and $X = G_n/\Lambda_n$ (solid curve). The approximations given by equations (57) and (59) are also shown in this figure by dotted curves. One can see that $Y(X)$ is a universal function which does not depend on the band number n in these normalized variables.

Now we consider the dependence of the gap on the zone number n for both $n \ll p_0$ and $n \gg p_0$. In figure 2(a) this dependence is shown for a superlattice with very small randomization when the conditions for the gaps to be open are satisfied for large enough values of n. In the case where $p_0 < 1$ the spectrum is described by equation (54) for all values of n. The circles in figure 2(a) correspond to this case: all open gaps from $n = 1$ to $n = 27$ satisfy the condition (61), which is valid for Gaussian correlations of the inhomogeneities. We chose the relation $G_1/\Lambda \equiv \sigma q^2/\Lambda = (\pi^{1/4}/\sqrt{2})(29)^{-3}$ for the calculation of this curve. Thus we find that the first closed gap corresponds to $n = 29$ in accordance with equation (59). The form of this curve does not depend on $k_\parallel$; that is why for the given relation $\sigma q^2/\Lambda$ it has the same form for all values $p_0 = k_\parallel/\sigma q < 1$.

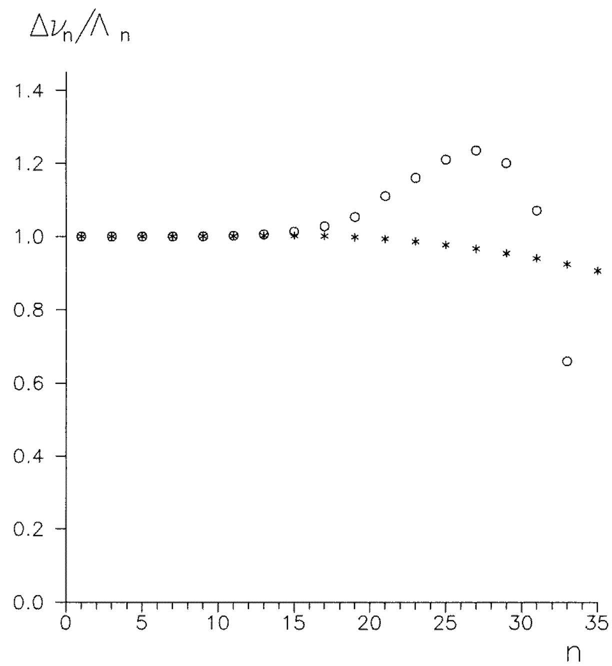
The stars in figure 2(a) correspond to the case where $p_0 \gg 30$. In this case where equation (47), which corresponds to the exponential correlations, is valid for all open gaps. In contrast to the preceding case, the form of the curve now depends on p_0. We calculated this curve for $\sigma q^2/\Lambda = (\pi^{1/4}/\sqrt{2})(29)^{-3}$ and $p_0 = 61$. The opening of new gaps for $n > 27$ in this case is determined by the fact that increasing p_0 decreases the damping parameter $G_2 = \sigma q^2/p_0$.

In this case the most important terms of the series in equation (43) have the form of $F_p^{(1)}$. This series can be summed approximately if we assume that all of the terms have the form of $F_p^{(1)}$ and neglect $(nk_c/q)^2$ in comparison with unity:

$$\nu - k^2 = \frac{\pi\Lambda^2}{8q\sqrt{\nu_1}}\left\{\frac{1}{\zeta_-}\left[1 - \left(\frac{1}{\Phi_-} - \frac{2\mathrm{i}k_c}{\pi q}\frac{\Phi_-}{\zeta_-}\right)\tan\Phi_-\right]\right.$$
$$\left. + \frac{1}{\zeta_+}\left[1 - \left(\frac{1}{\Phi_+} - \frac{2\mathrm{i}k_c}{\pi q}\frac{\Phi_+}{\zeta_+}\right)\tan\Phi_+\right]\right\} \tag{62}$$



(a)



(b)

Figure 2. The dependence of the normalized width of the gap $\Delta\nu_n/\Lambda_n$ on the zone number n for one-dimensional (a) and three-dimensional (b) inhomogeneities. Stars correspond to the exponential correlations of one-dimensional inhomogeneities, and the power-like correlations of three-dimensional inhomogeneities; circles correspond to the Gaussian correlations for all dimensions.

where

$$\Phi_{\pm}^2 = \left(\frac{\pi}{2}\right)^2 \frac{(\sqrt{\nu}_1 \pm k_z)^2}{q^2 + 2\mathrm{i}k_{c2}(\sqrt{\nu}_1 \pm k_z)}.$$

When $k_{c2} \to 0$, this equation transforms into equation (31), which describes the exact Bourret approximation in the absent of the randomization. Equation (62) can be useful if it is necessary to find corrections to the two-wave approximation described by equation (47).

4.2. *Isotropic three-dimensional inhomogeneities*

Substituting equations (41) and (42) into equation (21) we obtain the equation for the spectrum in the same general form as equation (43), but with different terms in the series:

$$\nu - k^2 = \frac{\Lambda^2}{4}\left\{\sum_{|p|\ll p_0} L_p^{(1)}(\nu, \boldsymbol{k}) + \sum_{|p|\gg p_0} L_p^{(2)}(\nu, \boldsymbol{k})\right\} \tag{63}$$

where $p_0 = k_0/\sigma q$. For the case where $|p| \ll p_0$ we obtain

$$\begin{aligned} L_p^{(1)} = &\left(\frac{1}{p^2} - \frac{1}{p_0^2}\right)\frac{1}{\nu - (\boldsymbol{k} - p\boldsymbol{q})^2} \\ &+ \frac{1}{2k_0 p_0^2|\boldsymbol{k} - p\boldsymbol{q}|}\left[\frac{1}{v_1 - \mathrm{i}} - \frac{1}{u_1 - \mathrm{i}} + 2\mathrm{i}\left(\ln\frac{u_1 - \mathrm{i}}{u_1} - \ln\frac{v_1 - \mathrm{i}}{v_1}\right)\right] \end{aligned} \tag{64}$$

where

$$\begin{aligned} u_1 &= (\sqrt{\nu} - |\boldsymbol{k} - p\boldsymbol{q}|)/k_0 \\ v_1 &= (\sqrt{\nu} + |\boldsymbol{k} - p\boldsymbol{q}|)/k_0. \end{aligned} \tag{65}$$

For the case $|p| \gg p_0$ we have

$$L_p^{(2)} = \frac{1}{\sqrt{2}k_{c3}p^3}\frac{1}{|\boldsymbol{k} - p\boldsymbol{q}|}\left[D(u_2) - D(v_2) + \mathrm{i}\frac{\sqrt{\pi}}{2}(\mathrm{e}^{-u_2^2} - \mathrm{e}^{-v_2^2})\right] \tag{66}$$

where

$$\begin{aligned} u_2 &= (\sqrt{\nu} - |\boldsymbol{k} - p\boldsymbol{q}|)/\sqrt{2}pk_{c3} \\ v_2 &= (\sqrt{\nu} + |\boldsymbol{k} - p\boldsymbol{q}|)/\sqrt{2}pk_{c3}. \end{aligned} \tag{67}$$

Just as in the case of the one-dimensional homogeneities treated above, for the three-dimensional inhomogeneities we can describe the spectrum in the vicinity of each crossing resonance $k_{rn} = nq/2$ in the two-wave approximation by using only the term of the series with $p = n$. For wave propagation along the z-axis we obtain in the case where $|n| \ll p_0$ the dispersion relation

$$\begin{aligned} \nu - k^2 = \frac{\Lambda^2}{4}&\left\{\left(\frac{1}{n^2} - \frac{1}{p_0^2}\right)\frac{1}{\nu - (\boldsymbol{k} - n\boldsymbol{q})^2}\right. \\ &\left.+ \frac{1}{2k_0 p_0^2|\boldsymbol{k} - n\boldsymbol{q}|}\left[\frac{1}{v_1 - \mathrm{i}} - \frac{1}{u_1 - \mathrm{i}} + 2\mathrm{i}\left(\ln\frac{u_1 - \mathrm{i}}{u_1} - \ln\frac{v_1 - \mathrm{i}}{v_1}\right)\right]\right\}. \end{aligned} \tag{68}$$

This equation can be investigated approximately in limiting cases. For the case where $|u| \gg 1$, $|v| \gg 1$ we obtain for the width of the gap and the damping parameters at the nth Brillouin zone boundary $k = k_{rn}$

$$\Delta\nu_n \approx \Lambda_n\left[1 + \frac{2}{3}\left(\frac{n}{p_0}\right)^2\eta_n\right] \tag{69}$$

$$\xi_{\pm} = \Lambda_n\left(\frac{n}{p_0}\right)^2\eta_n^3 \tag{70}$$

where $\eta_n = k_0 q n/\Lambda_n \ll 1$.

In another limiting case, that where $|u| \ll 1$, $|v| \gg 1$, when $\eta_n \gg 1$ we have

$$\Delta\nu_n \approx \Lambda_n\left[1 - \frac{1}{2}\left(\frac{n}{p_0}\right)^2\left(1 - \frac{\pi}{2\eta_n}\right)\right] \tag{71}$$

$$\xi_\pm = \frac{\Lambda_n}{4\eta_n}\left(\frac{n}{p_0}\right)^2 \ln(2\eta_n). \tag{72}$$

One can see that equation (71) for $n/p_0 \ll 1$ does not describe the situation when $\Delta\nu \to 0$. In contrast to the one-dimensional case, the gap cannot close under the influence of three-dimensional inhomogeneities until the inequality $n \ll p_0$ is satisfied.

We now turn to the terms of the series (63) corresponding to $|p| \gg p_0$. For the waves propagating along the z-axis the equation for the spectrum in the two-wave approximation has the form

$$\nu - k^2 = \frac{q\Lambda^2}{4\sqrt{2}G_3 n^3}\frac{1}{|\boldsymbol{k} - n\boldsymbol{q}|}\left[D(u_2) - D(v_2) + \mathrm{i}\frac{\sqrt{\pi}}{2}(\mathrm{e}^{-u_2^2} - \mathrm{e}^{-v_2^2})\right] \tag{73}$$

where $G_3 = k_{c3}q = \sigma q^2/\sqrt{3}$. At first glance it would seem that this equation has significant differences from equation (54), which corresponds to one-dimensional inhomogeneities: the functions of v have opposite signs, and the quantity $|\boldsymbol{k} - n\boldsymbol{q}|$ occurs in the denominator of the right-hand side instead of $\sqrt{\nu}$. However, numerical calculations demonstrate that the solutions of equation (73) differ little from the solutions of equation (54) investigated above. An analytical analysis of the limiting cases of $G_{n3} \ll \Lambda_n$ and $G_{n3} \sim \Lambda_n$ gives the same equations (57), (58) and (59), (60), respectively, which have been obtained for equation (54), with the natural change of the damping parameter $G_1 = k_{c1}q$ in all of these expressions to the damping parameter $G_3 = k_{c3}q$ corresponding to equation (73). The relation between these parameters is $G_3/G_1 = k_{c3}/k_{c1} = 1/\sqrt{3}$ and, for example, for the damping at the nth Brillouin zone boundary we obtain

$$\xi_\pm = n^2\sigma q^2/\sqrt{6}. \tag{74}$$

The dependence of $\Delta\nu_n$ on the zone number n is shown in figure 2(b). Both curves have been calculated for the relation $\sigma q^2/\Lambda = (\pi^{1/4}/\sqrt{2})(29)^{-3}$, as in the one-dimensional case. As in figure 2(a) the circles correspond to the case where $p_0 < 1$, and the stars correspond to the case where $p_0 = 61$. Comparing the results for one- and three-dimensional inhomogeneities one can see that for the case where $p_0 < 1$ corresponding to smooth inhomogeneities with Gaussian correlations new gaps corresponding to $n = 29, 31$, and 33 open in the three-dimensional case in accordance with equation (59), where the damping parameter G_1 has been replaced by the smaller parameter G_3. Even greater differences between the one- and three-dimensional cases are found for the short-wavelength inhomogeneities when $p_0 = 61$ (we assume that $k_0 = k_\parallel$). Many new gaps open in the three-dimensional case when the random deformation of the interfaces is added to their random displacements, the only form of randomization considered in the one-dimensional case.

Both figures 2(a) and 2(b) have an illustrative nature. A system for which about 30 Brillouin zones with open gaps could be investigated is far from reality. But for a real system with only several open gaps the dependence of $\Delta\nu_n/\Lambda_n$ on n will be the same as in figures 2(a), 2(b), only the points will be plotted very sparsely.

In comparing the results for one- and three-dimensional inhomogeneities one should take account of the fact that the analytical expressions for the cases of $G_{in} \ll \Lambda_n$ and $G_{in} \sim \Lambda_n$ are different. We compare here the expression for the effective damping parameter $\xi_\pm$ at the nth Brillouin zone boundary for the cases where $G_{in} \sim \Lambda_n$, $i = 1, 2, 3$. One can see that for the case where $n \gg p_0$ equation (60) for one-dimensional and equation (74) for three-dimensional inhomogeneities differ from one another only by numerical coefficients (the damping for three-dimensional inhomogeneities is smaller than that for one-dimensional inhomogeneities). At the same time, for the case where $n \ll p_0$ equation (51) for one-dimensional and equation (72) for three-dimensional inhomogeneities have different dependences on the fundamental characteristic of the superlattice (Λ and q), the inhomogeneities (σ and k_i), and the zone number n.

The expressions (51), (58), (60), (70), (72), and (74) give the damping parameters in units of the square of a wave vector. To obtain the quantities corresponding to the imaginary value of the frequency ω'' these parameters have to be multiplied by $\alpha g M$ for spin waves, by $v/2k$ for elastic waves, and by $c/2k\sqrt{\varepsilon_e}$ for electromagnetic waves.

5. Conclusions

The approach to the investigation of the wave spectrum of partially randomized superlattices that was suggested in reference [15] has been extended here to the case of superlattices with sharp interfaces, i.e. multilayer structures. The dependence of the material parameters on the coordinate along the axis of the initial ideal superlattice now has the form of periodic rectangular pulses which we represent by their Fourier series. The randomization of the superlattice is described by introducing a random modulation of the period of the initial ideal superlattice. One- and three-dimensional modulations are considered. As in reference [15], the spectrum and damping of the wave is investigated in the Bourret approximation, which corresponds to taking into account the first term of the series for the mass operator of the averaged Green's function. For the harmonic superlattice this approximation permits investigating only the first Brillouin zone, because the spectrum of the zones with $n \neq 1$ is determined by the next terms of the series.

In contrast to this, the Bourret approximation for the multilayer structure gives the possibility of investigating the spectrum and damping in the vicinity of the boundary of any odd Brillouin zone. Because of this, the gap widths and the values of the damping parameters are found for all odd Brillouin zones, and the dependences of these characteristics on the zone number n are found. As for superlattices with initial harmonic dependences of their material parameters, for superlattices with sharp interfaces different results are obtained for short-wavelength and smooth inhomogeneities. But the demarcation line between smooth and short-wavelength inhomogeneities depends now on the zone number n. The inhomogeneities characterized by the intensity σ and the correlation wavenumber $k_\parallel$ (for the one-dimensional case) or k_0 (for three-dimensional case) are the short-wavelength ones for the Brillouin zones with $n < k_\parallel/\sigma q$ and the smooth ones for the zones with $n > k_\parallel/\sigma q$. It was found that the damping parameters and the conditions for the closing of the gaps depend differently on the zone number n for the short-wavelength and smooth inhomogeneities. There are significant differences in the dependences of the gap width on n for the one- and three-dimensional inhomogeneities, especially for the short-wavelength ones. The appearance of the random deformation of the interfaces along with their random displacements from the initial positions leads to a decrease of the damping and to the opening of new gaps in comparison with the one-dimensional case, where only random displacements of interfaces occur. In all cases, with increasing disorder the successive closing of the gaps in the spectrum takes place beginning with large values of n down to $n = 1$.

Experimental investigations of the spin-wave spectrum are restricted for the present to the vicinity of the first Brillouin zone boundary [24, 25]. It would be of interest to carry out experiments covering several Brillouin zones to investigate the regularities described by the equations of this paper.

Acknowledgments

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